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MINIMAL SEIFERT MANIFOLDS AND THE KNOT FINITENESS THEOREM

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ABSTRACT

It is proved that the Alexander modules determine the stable type of a knot up to finite ambiguity. The proof uses a new existence theorem of minimal Seifert surfaces for multidimensional knots of codimension two.

Introduction

In 1930 F. Frankl and L. Pontryagin [FP] proved that any polygonal simple closed curve in three-dimensional space spans an orientable nonsingular surface in R^3. This statement served as the starting point for Seifert's paper [S], released three years later, which created a new method of studying algebraic invariants of classical links. Seifert's idea was to use the information, derived from the imbedding in R^3 of the surface spanned by the link, to compute link invariants, such as homology of the cyclic coverings. This made it possible to apply homology methods in knot theory and, in particular, to explain the homological meaning of many knot invariants known at that time.

Seifert's method proved to be extraordinarily useful in multidimensional knot theory as well. In 1965–1966 M. A. Kervaire [K4], E. C. Zeeman [Z] and J. P. Levine [L1] showed that any (smooth) n-dimensional knot $k^n \subset S^{n+2}$ spans an orientable $(n+1)$-dimensional submanifold $V^{n+1} \subset S^{n+2}$, called a Seifert manifold. The embedded surgery technique and the study of the

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homology pairing, determined by linking numbers of cycles of V (the Seifert pairing), were the main tools used to solve a number of fundamental problems of mutlidimensional knot theory. In this way the unknotting criterion [L1], the concordance classification of knots [L3] and the ambient isotopy classification of simple knots [L4], [F3] were obtained.

The method used in the papers mentioned above consisted, essentially, in the transformation of the Seifert manifold V inside S^{n+2} in order to get another Seifert manifold, having "simpler" homology structure. It is natural to ask if there exist "simplest" or "minimal" Seifert manifolds? For the case of a fibred knot the answer to this question is clear: a minimal Seifert manifold should be the fibre of the corresponding fibering.

The present paper studies minimal Seifert manifolds of general multidimensional knots. The formal definition of minimality is given in 2.4; it is equivalent to the requirement that the inclusion of int V into the infinite cyclic covering $\tilde{X}$ induces a monomorphism in homology. This means the absence of "superfluous" homology classes in V: any class in the kernel of $H_* V \to H_* \tilde{X}$ could potentially be killed by surgery, and thus V can be made "smaller". One of the main theorems of the present paper (Theorem 2.3) states that any knot (S^{n+2}, k^n) with $\pi_1(S^{n+2} - k) = Z$, $n \geqq 4$, has a minimal Seifert manifold V, and this V can be constructed to realize any previously given sequence of lattices in the Alexander modules. In other words, the homology structure of a minimal V might be, to some extent, arbitrary; the only requirement is that it should be compatible with the Alexander modules. This statement is a far-reaching generalization of the well-known Levine's theorem [L], which provides r-connected Seifert manifolds. Another slightly more general statement, which also follows from our Theorem 2.3, yields the following lacunary principle: if the Alexander modules vanish in certain dimensions $i_1 < i_2 < \cdots < i_l$, then the knot admits a Seifert manifold without homology in dimensions $i_1 < i_2 < \cdots < i_l$ (cf. Corollary 2.7).

As the first application of the minimal Seifert manifolds theorem, I give here a very short proof of the Trotter–Kearton theorem [Tr], [K3], saying that simple odd-dimensional knots are equivalent if and only if they have congruent Milnor forms. The second application is the knot finiteness theorem, proved in Section 3. This theorem states that, modulo finiteness, the stable type of a knot is determined by the Alexander modules up to the middle dimension. A particular case of this statement was proved by Haussmann [H1].

Theorem 2.3 on minimal Seifert manifolds is deduced in the paper from a certain general realization theorem for codimension-one submanifolds. As is well known, any one-dimensional integer cohomology class can be realized by a submanifold with trivial normal bundle [Th]. The question considered in Section 1 of this paper is if it is possible to construct the *minimal* realizing submanifold and describe the whole variety of such minimal submanifolds. The answer is given by Theorem 1.5: minimal submanifolds correspond to sequences of lattices in the homology modules of the covering space. This statement constitutes, essentially, the main geometric part of the proof [F6] of exactness of the Novikov inequalities [N1], [N2], estimating the number of critical points of a map into the circle. Theorem 1.5 (or the theorem of [F6]) easily implies the theorem of Browder and Levine [BL] on fiberings over S^1.

An earlier version of the results of this paper was announced in the brief note [F5].

In the sequel the exposition is organized as follows. The realization theorem for codimension one submanifolds and its proof are given in Section 1; proofs of some lemmas, used here, are placed in a separate Section 5 at the end of the article. Section 2 gives the conceptual background and the formulation of the theorem on minimal Seifert manifolds; here we also deduce some of its easy corollaries. Section 3 is devoted to the applications: the Kearton–Trotter and the knot finiteness theorem are proved here. The following Section 4 contains auxiliary algebraic material used in the proof of Theorem 2.3; here the proof of this theorem is also given. Among the results of this section I will mention a new simple construction of the Milnor pairing, cf. [M].

We work in the smooth category.

I would like to thank Eva Bayer-Fluckiger for useful discussions.

§1. Realization theorem for codimension one submanifolds

In this section we will formulate a realization theorem for codimension one submanifolds. The proof uses several lemmas, which will be proved in Section 5.

1.1. *Modules Defined by a Codimension One Submanifold.* Let M^n be a compact connected manifold and (V^{n-1}, ν) its framed proper smooth submanifold. Let us cut M along V^{n-1} ([BL], 2.2); as a result we get a compact manifold Y (with corners), in whose boundary two disjoint $(n-1)$-dimensional submanifolds $V_0, V_1 \subset \partial Y$ are distinguished, and a quotient mapping $\psi: Y \to M$ with the following properties:

(1) for $m \in M - V$ the preimage $\psi^{-1}(m)$ consists of one point;

(2) for $m \in V$ the preimage $\psi^{-1}(m)$ consists of two points, one belonging to V_0 and the other to V_1, and

(3) ψ maps V_0 and V_1 homeomorphically onto V.

We shall assume that notations V_0 and V_1 are chosen so that the vector field on V_0 corresponding to v under ψ is directed into Y.

Let a manifold $X = X(V, v)$ be obtained from $Y \times N$ (where N is the set of natural numbers with the discrete topology) by identifying points (v_1, m) with $(v_0, m+1)$ where $m \in N$, $v_1 \in V_1$, $v_0 \in V_0$ and $\psi(v_1) = \psi(v_0)$. We denote by $q: X \to M$ the unique projection mapping the class of a point $(y, m) \in Y \times N$ into $\psi(y)$. The correspondence $(y, m) \to (y, m+1)$ defines a continuous mapping $X \to X$ which we shall denote by t. The group $H_*(X)$ becomes a Λ_+-module, where $\Lambda_+ = Z[t]$, if one puts $tx = t_*(x)$ for $x \in H_*(X)$. In this way a sequence of Λ_+-modules $A_i(V, v) = H_i(X(V, v))$, $i = 0, 1, \ldots$, is defined.

1.2. *Modules Defined by a Cohomology Class.* Let $\xi \in H^1(M; Z)$ be an indivisible cohomology class. Consider the covering $p_\xi: M_\xi \to M$ corresponding to a subgroup in $\pi_1(M)$ consisting of classes of loops α for which $(\xi, \alpha) = 0$. The group of covering transformations of this covering is an infinite cyclic group. Its generator $t: M_\xi \to M_\xi$ can be fixed by requiring that for $x \in M_\xi$ and for any path ω in M_ξ joining x with tx, the value of the class ξ on the homology class of the loop $[p \cdot \omega] \in \pi_1(M)$ be equal $+1$. The homology $H_*(M_\xi)$ are Λ-modules, where $\Lambda = Z[t, t^{-1}]$

1.3. *The Embedding $\mu: X(V, v) \to M_\xi$.* Suppose that in the situation of Subsection 1.1 it is known that the manifold V^{n-1} is connected and that the cohomology class $\xi = \theta(V, v) \in H^1(M; Z)$, which is realized by the submanifold (V, v), is nonzero.

Then the class ξ is indivisible (and therefore the arguments of the previous subsection are applicable to it).

From the theory of covering spaces it easily follows that the map $q: X(V, v) \to M$, defined is 1.1, admits a lifting $\mu: X(V, v) \to M_\xi$, which is an equivariant imbedding.

1.4. Let $k \subset K$ be two neotherian rings, and A be a finitely generated K-module. A k-submodule $S \subset A$ is called a k-lattice if it is finitely generated (over k) and generates the module A over the ring K.

Let us consider the map $\mu_*: A_i(V, v) \to H_i(M_\xi)$, which is induced by the map $\mu: X(V, v) \to M_\xi$ from Subsection 1.3. It is easy to see that μ_* is a

Λ_+-homomorphism and that $B_i(V, v) = \text{image}(\mu_*)$ is a Λ_+-lattice of the Λ-module $H_i(M_\xi)$.

1.5. THEOREM (The Realization Theorem). *Let $n \geqq 6$, M^n be a compact connected manifold with $\pi_1 M = Z$, and $\xi \in H^1(M; Z)$ be a generator. Let us suppose that a framed smooth compact submanifold (F^{n-1}, v_0) in ∂M is given and for each $i = 2, 3, \ldots, n-3$ in the module $H_i(M_\xi)$ some Λ_+-lattice C_i is distinguished. It is assumed that the following conditions are satisfied:* (a) $\xi \mid_{\partial M} = \theta(F, v_0)$; (b) *there exists a smooth fibering $g: \partial M \to S^1$, realizing the class $\xi \mid_{\partial M}$, such that for some point $s \in S$ we have $g^{-1}(s) = F$. Then there exists a compact simply connected smooth framed proper submanifold (V^{n-1}, v) in M such that:*

(I) $\theta(V, v) = \xi$;

(II) $\partial V = F$, $v \mid_F = v_0$;

for each $i = 2, 3, \ldots, n-3$ the following conditions hold:

(III$_i$) *the homomorphism $\mu_*: A_i(V, v) \to H_i(M_\xi)$ is a monomorphism;*

(IV$_i$) $B_i(V, v) = t^{\alpha_i} C_i$ *for some integer $\alpha_i \in Z$.*

The proof (see 1.9 below) will be obtained by constructing a process of improving manifold (V, v), based on the following lemmas.

1.6. LEMMA. *Suppose that under the conditions of Theorem 1.5 for some integer k, $2 \leqq k \leqq n-3$, we have constructed a compact, simply connected, smooth, framed, proper, submanifold (V^{n-1}, v) in M which satisfies the conditions* (I), (II) *and also conditions* (III$_i$) *and* (IV$_i$) *for all $i < k$. Then there exists a framed submanifold $(W^{n-1}, \omega) \subset M^n$, which satisfies condition* (III$_k$) *in addition to the above conditions and moreover $B_k(W, \omega) = t^{\alpha_k} B_k(V, v)$ for some integer α_k.*

1.7. LEMMA. *Suppose that under the assumptions of Theorem 1.5 for some integer k, $2 \leqq k \leqq n-3$, we have constructed a compact, simply connected, smooth framed, proper, submanifold (V^{n-1}, v) in M, satisfying (in the notations of the Theorem 1.5) conditions* (I), (II), (III$_i$) *for all $i \leqq k$, and the conditions* (IV$_i$) *for all $i \leqq k-1$. Suppose also that $tB_k(V, v) \subset C_k \subset B_k(V, v)$ and the factor group $C_k / tB_k(V, v)$ is cyclic. Then there exists a framed submanifold $(W^{n-1}, \omega) \subset M^n$, which, besides the conditions listed above, satisfies also condition* (IV$_k$).

In the proof of Theorem 1.5 we shall also use the following purely algebraic lemma.

1.8. Lemma. *Let H be a finitely generated Λ-module and let $B, C \subset H$ be two of its Λ_+-lattices. Then there exist an integer $\alpha \geqq 0$ and a sequence of Λ_+-lattices $A_0, A_1, \ldots, A_N \subset H$, such that $A_0 = B$, $A_N = t^\alpha C$, for $i = 0, 1, 2, \ldots, N-1$ the following inclusions hold: $tA_i \subset A_{i+1} \subset A_i$ and the factor groups A_{i+1}/tA_i are cyclic.*

1.9. Proof of Theorem 1.5. Using the Pontryagin–Thom construction, one may construct a proper, framed submanifold $(V^{n-1}, \nu) \subset M^n$, satisfying conditions (I) and (II) of Theorem 1.5 (see [BL], 2.1). As shown in Subsections 3.1 and 3.2 of [BL], under the conditions of Theorem 1.5 there exists a submanifold (V^{n-1}, ν) which, besides the above listed conditions, is simply connected (and, in particular, connected). This simply connected submanifold will serve as the beginning of the inductive process.

By Lemma 1.6, we may assume that condition (III_2) is also satisfied. By Lemma 1.7 (combined with Lemma 1.8) we may suppose that condition (IV_2) is also satisfied. Then we may apply Lemma 1.6 again to get conditions (I), (II), (III_2), (III_3), and (IV_2) and, applying Lemmas 1.7 and 1.8, we shall get a simply connected, proper, framed submanifold of M^n, satisfying conditions (I), (II), (III_2), (III_3), (IV_2), (IV_3). Continuing this construction, we get the desired submanifold (V^{n-1}, ν).

This completes the proof. Lemmas 1.6, 1.7, 1.8 will be proved in §5.

§2. Minimal Seifert manifolds

Here we apply the realization Theorem 1.5 to the problem of constructing minimal Seifert surfaces for multidimensional knots.

Before giving the statement of the theorem we recall some definitions and known facts on the relationship between Alexander modules and the homology of Seifert manifolds.

2.1. An *n-dimensional knot* is a pair (S^{n+2}, k^n) consisting of the sphere S^{n+2} and of an n-dimensional closed oriented submanifold k of it that is homeomorphic (but not necessarily diffeomorphic) to the n-dimensional sphere S^n. A *Seifert manifold* of a knot (S^{n+2}, k^n) is any compact connected orientable $(n+1)$-dimensional submanifold $V \subset S^{n+2}$ with $\partial V = k$.

Let $K = (S^{n+2}, k^n)$ be an n-dimensional knot and $X = S^{n+2} - k$ is its complement. The universal abelian cover $p: \tilde{X} \to X$ is the covering projection corresponding to the commutator subgroup of $\pi_1(X)$. The group of covering transformations of p is the Abelianized group $\pi/[\pi, \pi] = H_1 X$. By the

Alexander duality theorem, $H_1 X = \mathbf{Z}$; hence, the cover $p : \tilde{X} \to X$ has an infinite cyclic group of covering transformations. The orientations of S^{n+2} and k determine a generator $t : \tilde{X} \to \tilde{X}$ of this group. t acts on homology $H_* \tilde{X}$, making it into modules over the ring $\Lambda = \mathbf{Z}[t, t^{-1}]$. The module $H_i \tilde{X}$ is called the i-dimensional *Alexander module* of K; we will denote it by $A_i(K)$.

Kervaire [K4] has proved that the Alexander modules have the following property: multiplication by $1 - t \in \Lambda$ is an automorphism of $A_r(K)$. Using this fact one can provide $A_i(K)$ with a $P = \mathbf{Z}[z]$-module structure by putting $za = (1-t)^{-1}a$ for $a \in H_i(\tilde{X})$. A P-submodule $S \subset A_i(K)$ is called P-lattice if it is finitely generated over P and generate $A_i(K)$ over Λ.

For n odd, $n = 2q - 1$, the Milnor form [M2]

$$[\ ,\ ] : A_q(K) \times A_q(K) \to Q$$

is defined. If $S \subset A_q(K)$ is a lattice, then $S^{\#} = \{a \in A_q(K); [a, x] \in \mathbf{Z}\}$ is also a lattice (cf. Section 4) which is called the *dual* of S. A lattice S is *self-dual* iff $S = S^{\#}$.

Any Seifert manifold V of K has a natural P-module structure on $H_*(V)$ (see [K5], [F3, pp. 66–68]). The inclusion int $V \to X$ may be lifted to $\tilde{X}$ and any such lifting f: int $V \to \tilde{X}$ gives a map

$$f_* : H_i V \to A_i(K).$$

2.2. It was proved in [F3, pp. 76–80, 91–92] that (1) f_* is a P-homomorphism; (2) its image is a P-lattice in $A_i(K)$; (3) the kernel of f_* consists of all elements $v \in H_i V$ with $(z\bar{z})^m v = 0$ for some $m \geqq 0$ (here $\bar{z}$ means $1 - z \in P$); (4) if n is odd, $n = 2q - 1$, then the middle dimensional P-lattice $\operatorname{im}[f_* : H_q V \to A_q(V)] \subset A_q(K)$ is self-dual.

The following theorem is the main result of this section.

2.3. THEOREM. *Let $K = (S^{n+2}, k^n)$ be an n-dimensional knot with $\pi_1(S^{n+2} - k) = \mathbf{Z}$, $n \geqq 4$. Assume that for any $r = 2, \dots, q = [(n+1)/2]$ a P-lattice $S_r \subset A_r(K)$ is distinguished; in the case of odd n it is required that the middle-dimensional lattice $S_q \subset A_q(K)$ is self-dual. Then there exists a simply-connected Seifert manifold $V^{n+1} \subset S^{n+2}$ of the knot K such that for all $r = 2, \dots, q$ the P-module $H_r(V)$ is isomorphic to S_r. Moreover, in the case of odd n there exists an isomorphism $\varphi_q : H_q(V) \to S_q$ with the property:*

$$[\varphi_q(v_1), \varphi_q(v_2)] = \langle v_1, v_2 \rangle.$$

Here [,] denotes the Milnor form and $\langle\ ,\ \rangle$ means the intersection number pairing on V.

The proof of this theorem will be given in 4.11.

Now we would like to formulate some of its corollaries.

A P-module B will be called *minimal* if the multiplication by $z\bar{z} \in P$ is a monomorphism $B \to B$.

2.4. Lemma. *Let $V^{n+1} \subset S^{n+2}$ be a Seifert manifold of a knot. The following conditions are equivalent:*

(a) *the P-modules $H_r(V)$ are minimal for all $r = 1, 2, \ldots, n$;*

(b) *the P-modules $H_r(V)$ are minimal for all $r = 1, 2, \ldots, q = [(n+1)/2]$;*

(c) *the maps $i_+, i_- : V \to S^{n+2} - V$, which are small shifts in the directions of positive and negative normal vector fields, respectively, induce monomorphisms in homology;*

(d) *the maps $f_* : H_r(V) \to A_r(K)$, defined in 2.1, are monomorphisms for all $r = 1, \ldots, n$;*

(e) *the multiplication by $z \in P$ is a monomorphism $H_r V \to H_r V$ for all $r = 1, 2, \ldots, n$.*

The proof of the lemma will be given in 2.8. Seifert manifolds having one of the equivalent properties (a)–(e) will be called *minimal*.

2.5. Corollary. *Any n-dimensional knot, $n \geqq 4$, having group $\mathbf{Z}$, admits a minimal Seifert manifold.*

This automatically follows from Theorem 2.3, since the P-modules S_r, $r = 1, \ldots, q$, being submodules of the Alexander modules, are minimal and so $H_r(V)$ are minimal for $r = 2, \ldots, q$.

As another corollary of Theorem 2.3 we obtain the following known result:

2.6. Corollary (Levine [L1]). *Any n-dimensional knot $K = (S^{n+2}, k^n)$ with $\pi_i(S^{k+2} - k^n) = \pi_i(S^1)$ for $i \leqq r$, $n \geqq 4$, admits an r-connected Seifert manifold.*

In this case the Alexander modules $A_i(K)$ with $i = 1, 2, \ldots, r$ vanish.

The next statement is slightly more general.

2.7. Corollary. *Assume that an n-dimensional knot $K = (S^{n+2}, k^n)$, $n \geqq 4$, having group $\mathbf{Z}$, has the following lacunary property: for some set of numbers $i_1 < i_2 < \cdots < i_l \leqq q = [(n+1)/2]$ the Alexander modules vanish: $A_{i_s}(K) = 0$ for $s = 1, \ldots, l$. Then the knot admits a Seifert surface V with $H_{i_s}(V) = 0$ for $s = 1, 2, \ldots, l$.*

It follows easily from 2.5.

2.8. Proof of Lemma 2.4. The equivalence (a)$\Leftrightarrow$(c) follows easily from the relations

$$i_+ = (i_+ - i_-) \circ z, \qquad -i_- = (i_+ - i_-) \circ \bar{z}$$

(cf. [F3], pp. 67–68) and the fact that $i_+ - i_-$ is a stable homotopy equivalence.

The equivalence (a)$\Leftrightarrow$(d) follows from [F3], Proposition 2.2.

To prove that (a)$\Leftrightarrow$(b), we consider two pairings:

$$\langle\ ,\ \rangle : H_r(V) \times H_{n+1-r}(V) \to \mathbf{Z},$$

$$\{\ ,\ \} : T_r(V) \times T_{n-r}(V) \to \mathbf{Q}/\mathbf{Z}.$$

The first is the intersection form, and the second is the linking form. T_i means the $\mathbf{Z}$-torsion subgroup of H_i. We will use the following properties of $\langle\ ,\ \rangle$ and $\{\ ,\ \}$:

(1) $\langle z\bar{z}a, b\rangle = \langle a, z\bar{z}b\rangle$ for $a \in H_r(V)$, $b \in H_{n+1-r}(V)$;

(2) $\{z\bar{z}a, b\} = \{a, z\bar{z}b\}$ for $a \in T_r(V)$, $b \in T_{n-r}(V)$;

(3) if $a \in H_r(V)$ and $\langle a, b\rangle = 0$ for all $b \in H_{n+1-r}(V)$ then $a \in T_r(V)$;

(4) if $a \in T_r(V)$ and $\{a, b\} = 0$ for all $b \in T_{n-r}(V)$ then $a = 0$.

Properties (3) and (4) are well-known, (1) follows from Proposition 1.2 of [F3], and (2) may be proved similarly.

Now suppose that V^{n+1} is a Seifert manifold with P-modules $H_r(V)$ minimal for $r = 1, \dots, q = [(n+1)/2]$. Suppose $a \in H_s(V)$ with $z\bar{z}a = 0$, $s > q$. Then for any $b \in H_{n+1-s}(V)$, because of the minimality of $H_{n+1-s}(V)$, there is $N > 0$ with $Nb = z\bar{z}b_1$ for some $b_1 \in H_{n+1-s}(V)$. Thus,

$$\langle a, b\rangle = \frac{1}{N}\langle a, Nb\rangle = \frac{1}{N}\langle a, z\bar{z}b_1\rangle$$

$$= \frac{1}{N}\langle z\bar{z}a, b_1\rangle = 0$$

and so $a \in T_s(V)$. Similarly, for any $c \in T_{n-s}(V)$, because of the minimality of $H_{n-s}(V)$, $c = z\bar{z}c_1$ for some $c_1 \in T_{n-s}(V)$, and so

$$\{a, c\} = \{a, z\bar{z}c_1\} = \{z\bar{z}a, c_1\} = 0.$$

Hence, $a = 0$.

This proves (b)$\Rightarrow$(a) and the converse (a)$\Rightarrow$(b) is evident.

The implication (e)$\Rightarrow$(b) might be proved similarly; (a)$\Rightarrow$(e) is evident, and the lemma follows.

§3. Applications: the Kearton–Trotter theorem and the finiteness theorem

Here we present two applications of the realization Theorem 2.3. As the first application we give a very short proof of the famous Kearton–Trotter theorem [K2], [K3], [Tr], which asserts that simple odd-dimensional knots are equivalent if and only if they have isomorphic Blanchfield or Milnor forms. Our proof is based upon Theorem 2.3 and Lemma 3 of J. Levine's paper [L4]. As the second application, we prove the assertion announced in [F2], [F5] that, up to a finite number of possibilities, stable knot type is determined by its dimension, the Alexander modules up to the middle dimension and the Blanchfield (or Milnor) pairing on the Alexander module of the middle dimension. For fibred knots this fact was established in [F1]. J. Haussmann [H1] has proved this result is a particular case of knots having only one non-zero Alexander module below the middle dimension.

The final version of our finiteness theorem (Theorem 3.6) uses algebraic results of [BKW] and [BKW1]; it states that we could delete the Blanchfield form from the collection of classifying invariants modulo finiteness in the odd-dimensional case also.

An odd-dimensional knot $K = (S^{n+2}, k^n)$, $n = 2q - 1$, is called *simple* [L4], if $\pi_1(S^{n+2} - k) = \mathbf{Z}$ and $A_q(K)$ is the only non-zero Alexander module.

3.1. THEOREM ([K2], [Tr]). *Assume $n = 2q - 1$, $q \geqq 3$, and K_1, K_2 are simple n-dimensional knots. Then K_1 is equivalent to K_2 if and only if there exists a Λ-isomorphism $f: A_q(K_1) \to A_q(K_2)$ preserving the Milnor (or Blanchfield) form.*

The last condition means that for $a, b \in A_q(K_1)$

$$[f(a), f(b)] = [a, b],$$

where [,] denotes the Milnor form. In [Tr] (or [F3]) it is shown that Milnor and Blanchfield forms mutually determine each other.

PROOF OF 3.1. Let V_1 be any minimal Seifert manifold of K_1. Denote by $S_1 \subset A_q(K_1)$ the P-lattice determined by V_1 via the construction described at the end of 2.1. Assume that $S_2 \subset A_q(K_2)$ is the image of S_1 under some isomorphism $A_q(K_1) \to A_q(K_2)$ preserving the Milnor form. By virtue of Theorem 2.3, we may construct a Seifert manifold V_2 of K_2, corresponding to S_2. Thus, we have P-isomorphism

$$H_q(V_1) \leftrightarrow H_q(V_2)$$

preserving the intersection numbers. Now V_1 and V_2 admit identical Seifert matrices (because the Seifert pairing θ might be expressed in the form $\theta(a \otimes b) = \langle a, zb \rangle$ through multiplication by $z \in P$ and the intersection form) and by Lemma 3 of [L4], V_1 and V_2 are ambient isotopic. So K_1 and K_2 are equivalent. This proves the theorem.

3.2. We will say that n-dimensional knots K_1 and K_2 are *homologically equivalent* (or have the same *homology type*) if there exist Λ-isomorphisms $f_r: A_r(K_1) \to A_r(K_2)$, $r = 1, 2, \ldots, q = [(n+1)/2]$; in the case of odd n we require that f_q preserves the Milnor form. We will also use the notion of stable equivalence: two knots are said to be *stably equivalent* if they become equivalent after application of an iteration of the Bredon suspension; details may be found in [B], [KN], [F7], [F8].

3.3. THEOREM. *There exist only a finite number of n-dimensional knots (for fixed n) which are all homologically equivalent but pairwise stably nonequivalent. In other words, the Alexander modules* $A_1(K), \ldots, A_q(K)$, *where* $q = [(n+1)/2]$, *together with the Milnor form* $A_q(K) \times A_q(K) \to \mathbf{Q}$ *(in the case of odd n) determine the stable type of an n-dimensional knot up to a finite ambiguity*:

PROOF. Assuming the contrary, suppose that there exists an infinite sequence $K_1, K_2, \ldots$ of n-dimensional knots which are all homology equivalent and pairwise stably nonequivalent. Our arguments will consist of three steps. In the first step we will show that we may assume all knots $K_1, K_2, \ldots$ to be stable. In the second stage we will prove that we may assume that all knots $K_1, K_2, \ldots$ admit Seifert manifolds of the same stable homotopy type; here the realization theorem 2.3 will be used. In the last stage, we will find a contradiction, using a study of the group of self-homotopy equivalences and its action on homology; this stage is very similar to the arguments in the fibred case [F1]. At this stage the general stable homotopy classification of knots [F3], [F4] will be used.

The first stage is the easiest. Taking $N > 0$ sufficiently large, consider the sequence

$$\omega^{2N}(K_1), \omega^{2N}(K_2), \ldots$$

where ω denotes the Bredon suspension [B], [KN]. All knots in this sequence are homology equivalent and pairwise stably nonequivalent. If N is large enough, for instance $2N > n$, then all knots $\omega^{2N}(K_j)$ are stable. So we will not lose generality if we assume all knots in the initial sequence

$$K_1, K_2, \ldots$$

to be stable.

Denote $q = [(n+1)/2]$. Let us distinguish a set of P-lattices $D_r \subset A_r(K_1)$, $r = 1, 2, \ldots, q$ in the Alexander modules of K_1; in the case of n odd, we require the middle dimensional lattice $D_q \subset A_q(K_1)$ to be self-dual. It is clear that it is possible to do this; we are able, for example, to consider the set of P-lattices determined by some minimal Seifert manifold of K_1, whose existence is guaranteed by Corollary 2.5.

Let

$$A_r(K_1) \to A_r(K_j), \qquad r = 1, 2, \ldots, q, \quad j = 1, 2, \ldots,$$

be the homomorphisms realizing the homology equivalence of knots K_1 and K_j. Denote by D_r^j the image of D_r under this isomorphism.

By Theorem 2.3 applied to the system of lattices $D_r^j \subset A_r(K_j)$, $r = 1, 2, \ldots, q$, there exists a Seifert manifold V_j of K_j such that the P-module $H_r(V_j)$ is isomorphic to D_r^j for $r = 1, 2, \ldots, q$, and the isomorphism $H_q(V_j) \to D_q^j$ takes the restriction of the Milnor form to the intersection form on V_j.

Thus, we have P-isomorphisms

$$\varphi_r^j : H_r(V_1) \to H_r(V_j), \qquad r = 1, 2, \ldots, q, \quad j = 1, 2, \ldots$$

where in the case of odd n, $n = 2q - 1$, the homomorphism

$$\varphi_q^j : H_q(V_1) \to H_q(V_j)$$

has the property

$$\langle \varphi_q^j(x), \varphi_q^j(y) \rangle_{V_j} = \langle x, y \rangle_{V_1},$$

for $x, y \in H_q(V_1)$, $\langle\ ,\ \rangle_{V_j}$ denoting the intersection form on V_j.

Define P-homomorphisms

$$\psi_r^j : B_r(V_1) \to B_r(V_j), \qquad r = 1, 2, \ldots, n$$

(B_r denoting the Betti group) in the following way: for $r \leqq q$, ψ_r^j is just the restriction of φ_r^j, and for $r > q$ we define ψ_r^j by requiring the condition

$$\langle \psi_r^j(x), \psi_r^j(y) \rangle_{V_j} = \langle x, y \rangle_{V_1} \tag{1}$$

for $x \in B_r(V_1)$, $y \in B_{n+1-r}(V_1)$. By non-singularity of $\langle\ ,\ \rangle$ this condition defines ψ_r^j uniquely. It is clear that ψ_r^j are P-isomorphisms and that (1) holds for all $r = 1, 2, \ldots, n$ (not only for $r > q$).

Manifolds $V_1, V_2, \ldots$ have isomorphic homology groups in all dimensions. For $r \leqq q$ this follows from the existence of the isomorphisms φ_r^j, while for $r > q$ the group $H_r(V_j)$ may be expressed as

$$\mathrm{Hom}(H_{n+1-r}(V_j); \mathbf{Z}) \oplus \mathrm{Ext}(H_{n-r}(V_j); \mathbf{Z})$$

(by Poincaré duality and the universal coefficients theorem), and now $n+1-r \leqq q$.

By Corollary 5.2 of [F1], in the sequence $V_1, V_2, \ldots$ there is an infinite subsequence consisting of stably homotopy equivalent spaces. Thus, we are able to assume that all spaces in the sequence $V_1, V_2, \ldots$ are stably homotopy equivalent. For each j let us fix some S-equivalence $\pi_j: V \to V_j$, where $V = V_1$, $\pi_1 = \mathrm{id}$.

Let $\mathbf{p}_j = (V_j, u_j, z_j)$ be the stable isometry structure of V_j. The set

$$\mathbf{q}_j = (V, u_j \circ (\pi_j \otimes \pi_j), \pi_j^{-1} \circ z_j \circ \pi_j)$$

is the stable isometry structure isomorphic to $\mathbf{p}_j$. Theorem 3.3 will follow if we can show that in the sequence

$$\mathbf{q}_1, \mathbf{q}_2, \ldots$$

at least two stable isometry structures are isomorphic (in fact, if $\mathbf{q}_i$ and $\mathbf{q}_j$ are isomorphic, then $\mathbf{p}_i$ and $\mathbf{p}_j$ are isomorphic and by Theorem 2.6 of [F4], K_i and K_j are equivalent — the contradiction).

Let us consider the set S_1 of all stable isometry structures $\mathbf{q} = (V, u, z)$ with $V = V_1$ fixed. On this set the group $G_1 = G^s(V)$ of S-equivalences $V \to V$ acts from the left: for $g \in G_1$ and $\mathbf{q} \in S_1$ we put

$$g\mathbf{q} = (V, u \circ (g^{-1} \otimes g^{-1}), g \circ z \circ g^{-1}).$$

Let us also consider the set S_2, whose elements are all collections $(p_1, \ldots, p_n; l_1, \ldots, l_n)$ where $p_r: B_r \to B_r$, $r = 1, 2, \ldots, n$ is a $\mathbf{Z}$-homomorphism, $B_r = B_r(V)$, and

$$l_r: B_r \times B_{n+1-r} \to \mathbf{Z}, \qquad r = 1, 2, \ldots, n$$

is a $\mathbf{Z}$-bilinear form.

Consider also the left action of the group

$$G_2 = \prod_{r=1}^{n} \mathrm{Aut}(B_r)$$

on S_2, which sends the pair, consisting of the sequence $\{g_r\}_{r=1}^n \in G_2$, $g_r \in \mathrm{Aut}(B_r)$ and the collection $(p_1, \ldots, p_n; l_1, \ldots, l_n) \in S_2$, into the element $(p'_1, \ldots, p'_n; l'_1, \ldots, l'_n) \in S_2$ with

$$p'_r = g_r \circ p_r \circ g_r^{-1}, \qquad l_r^1 = l_r \circ (g_r^{-1} \times g_{n+1-r}^{-1}).$$

There is the natural map $S_1 \to S_2$ which assigns to any stable isometry structure $\mathbf{q} = (V, u, z)$ the collection $(p_1, \ldots, p_n; l_1, \ldots, l_n)$ with $p_r : B_r \to B_r$ being the homomorphism induced by $z : V \to V$, and with $l_r : B_r \times B_{n+1-r} \to \mathbf{Z}$ being the homomorphism induced by $u : V \otimes V \to S^{n+1}$. We also have the natural map

$$G_1 \to G_2$$

sending any stable equivalence $g : V \to V$ to the sequence

$$\{g_r\}_{r=1}^n \in G_2 = \prod_{r=1}^n \mathrm{Aut}(B_r)$$

with $g_r : B_r \to B_r$ being the map induced by g.

The constructed sets S_1 and S_2 with G_1 and G_2 actions, respectively, satisfy all conditions of Proposition 5.4 of [F1]. (This follows from Propositions 5.1 and 5.3 of [F1].) By Proposition 5.4 we are able to state that the natural map

$$S_1/G_1 \to S_2/G_2$$

has the following property: the preimage of any finite set is finite.

The sequence of stable isometry structures

$$\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots,$$

constructed above, defines a sequence of elements of S_1, which lie in the different orbits of the action of G_1. Thus, we will get a contradiction if we show that the elements of S_2 corresponding to $\mathbf{q}_1, \mathbf{q}_2, \ldots$ lie in the same orbit of G_2.

The collection

$$(p_1^j, p_2^j, \ldots, p_n^j; l_1^j, \ldots, l_n^j) \in S_2,$$

corresponding to $\mathbf{q}_j$, can be defined by the following compositions:

$$p_r^j : B_r \xrightarrow{\pi_{j*}} B_r(V_j) \xrightarrow{(z_j)_*} B_r(V_j) \xrightarrow{(\pi_j^{-1})_*} B_r,$$

$$l_r^j : B_r \times B_{n+1-r} \xrightarrow{\pi_{j*} \times \pi_{j*}} B_r(V_j) \times B_{n+1-r}(V_j) \xrightarrow{\langle\, ,\, \rangle_{V_j}} \mathbf{Z},$$

for $r = 1, 2, \ldots, n$. For any $j = 1, 2, \ldots$ let us define the collection $(g_1^j, g_2^j, \ldots, g_n^j) \in G_2$ in the following way:

$$g_r^j : B_r \xrightarrow{\pi_{j*}} B_r(V_j) \xrightarrow{(\psi_r^j)^{-1}} B_r.$$

It is easy to verify that

$$g_r^j \circ p_r^j \circ (g_r^j)^{-1} = z_{1*} = p_r^1,$$

$$l_r^j \circ ((g_r^j) \times (g_{n+r-1}^j)^{-1}) = l_r^1,$$

for $r = 1, 2, \ldots, n$. This means that all collections $(p_1^j, p_2^j, \ldots, p_n^j; l_1^j, \ldots, l_n^j)$, $j = 1, 2, \ldots$, lie in the same orbit of G_2.

The theorem follows.

The next statement is equivalent to Theorem 3.3 via results of [F7], [F8], saying that for stable knots the notions of stable type and isotopy type coincide.

3.4. THEOREM. *For any fixed $n \geqq 5$ there exist only a finite number of stable n-dimensional knots, which are all homology equivalent but pairwise nonequivalent.*

3.5. It is proved in [BKW], [BKW1] that on a given Alexander module there exist only finitely many congruence classes of Blanchfield forms. Combining this result with Theorem 3.4 we get the following statement, which may be considered as the main result of this section.

3.6. THEOREM. *The set of Alexander modules $A_1, A_2, \ldots, A_q$, determines the type of a stable n-dimensional knot ($n = 2q$ or $n = 2q - 1$) modulo finiteness.*

Another algebraic finiteness theorem of E. Bayer and F. Michel [BM] states that there are finitely many isomorphism classes of Alexander modules with given square-free Alexander polynomial. From this and from Theorem 3.6 there follows:

3.7. THEOREM. *Given integer n and a set of square-free integral polynomials $\Delta_1, \Delta_2, \ldots, \Delta_q$, where $q = [(n+1)/2]$, there exists only a finite number of types of stable n-dimensional knots having Δ_i as i-dimensional Alexander polynomial for $i = 1, 2, \ldots, q$.*

§4. Knot modules: lattices and duality

In this section auxiliary algebraic facts are presented, on which the proof of Theorem 2.3 is based. Our main interest is in the study of the relations between P- and Λ_+-lattices of knot modules, in the light of the Milnor duality. In fact, we deduce Theorem 2.3 from the general realization Theorem 1.5 by showing that the sequence of Λ_+-lattices in Alexander modules of all dimensions, corresponding to a Seifert manifold, is completely determined by the sequences of P-lattices in Alexander modules below the middle dimension, and vice versa.

Recall some of our notations: $\Lambda = \mathbf{Z}[t, t^{-1}]$, $\Lambda_+ = \mathbf{Z}[t]$, $\Lambda_- = \mathbf{Z}[t^{-1}]$, $L = \mathbf{Z}[t, t^{-1}, (1-t)^{-1}]$, $P = \mathbf{Z}[z]$. We consider P as a subring in L, identifying z with $(1-t)^{-1}$. We also may write $L = \mathbf{Z}[z, z^{-1}, \bar{z}^{-1}]$ where $\bar{z}$ always means $1-z$.

A *module of type K* [L5] is a finitely generated Λ-module A with the property that the multiplication by $1 - t \in \Lambda$ is an automorphism $A \to A$. The Alexander modules of knots of codimension two are of type K.

4.1. Proposition. *Let A be a module of type K and $R \subset A$ be a P-lattice. Then*

(1) *R is finitely generated as an abelian group*;

(2) *any element $a \in A$ can be represented in the form $a = (z\bar{z})^{-n} b$ for some $b \in R$, $n \geqq 0$*;

(3) $\mathrm{Tors}_{\mathbf{Z}} A \subset R$;

(4) *for any $a \in A$ there exists an integer $N \neq 0$ with $Na \in R$*;

(5) *the Λ_+-lattice $\Lambda_+ R$, generated by R, coincides with*

$$\{a \in A;\ z^n a \in A \text{ for some } n \geqq 0\};$$

(6) *the Λ_--lattice $\Lambda_- R$, generated by R, coincides with*

$$\{a \in A;\ \bar{z}^n a \in A \text{ for some } n \geqq 0\};$$

(7) $R = \Lambda_+ R \cap \Lambda_- R$;

(8) *$t^k \Lambda_+ R \cap t^l \Lambda_- R = \bar{z}^k z^{-l} R$, where k and l are integers.*

Proof. (1) It is well known [L5] that there exists an integer polynomial $\Delta(t) \in \Lambda_+$ with $\Delta(1) = 1$ and $\Delta A = 0$. Let $\Delta(t)$ be

$$\Delta(t) = 1 + \alpha_1(1-t) + \cdots + \alpha_k(1-t)^k,$$

with $\alpha_i \in \mathbf{Z}$. Define

$$\nabla(z) = z^k + \alpha_1 z^{k-1} + \cdots + \alpha_k.$$

Then $\nabla(z)a = 0$ for all $a \in A$. So, if elements $a_1, \ldots, a_l \in R$ generate R as a P-module, then elements

$$a_1, \ldots, a_l, za_1, \ldots, za_l, \ldots, z^{k-1}a_1, \ldots, z^{k-1}a_l$$

generate R over $\mathbf{Z}$. This proves (1).

(2) follows from the fact that the ring L is a localization of P over the multiplicative set, consisting of powers $(z\bar{z})^n$, $n \geqq 0$.

(3) The multiplication by $z\bar{z} \in P$ is a monomorphism $R \to R$. By virtue of (1) the group $T = \text{Tors}_{\mathbf{Z}} R$ is finite, and so the multiplication by $z\bar{z}$ is an isomorphism $T \to T$.

Assume $a \in \text{Tors}_{\mathbf{Z}} A$. By (2), $a = (z\bar{z})^{-n}b$ for some $b \in R$, $n \geqq 0$. Since a is of finite order, b is of finite order too, i.e., $b \in T$. From the previous paragraph we know that we may write $b = (z\bar{z})^n b_1$ for some $b_1 \in T$. Then $a = (z\bar{z})^{-n}b = b_1$ and so $a \in T$.

(4) By virtue of (1), there exists an integer $N \neq 0$ such that $NR \subset (z\bar{z})R$. If $a \in A$, $a = (z\bar{z})^{-n}b$ for some $n \geqq 0$, $b \in R$, then $N^n a \in R$.

(5) If $a \in \Lambda_+ R$ and $e_1, \ldots, e_m \in R$ generate R over $\mathbf{Z}$, then

$$a = \lambda_1(t)e_1 + \cdots + \lambda_m(t)e_m,$$

where $\lambda_i(t) \in \Lambda_+$, $i = 1, 2, \ldots, m$. Thus,

$$z^n a = z^n\lambda_1(t)e_1 + \cdots + z^n\lambda_m(t)e_m,$$

and for large enough $n \geqq 0$ all $z^n\lambda_i(t)$ might be written as polynomials in z (since $zt = z - 1$). So, $z^n a \in R$.

Conversely, assume $z^n a \in R$. Then

$$a = z^{-n}(\mu_1 e_1 + \cdots + \mu_m e_m), \qquad \mu_i \in \mathbf{Z},$$

and since

$$z^{-n} = (1 - t)^n \in \Lambda_+,$$

we have $a \in \Lambda_+ R$. (6) can be proved similarly.

(7) If $a \in \Lambda_+ R \cap \Lambda_- R$, then for large enough $n \geqq 0$

$$z^n a \in R, \qquad \bar{z}^n a \in R.$$

One can find an integral polynomial $f(z)$ with the property

$$z^n f(z) + \bar{z}^n f(\bar{z}) = 1,$$

for example

$$f(z)=\sum_{k=0}^{n-1}\binom{k}{2n-1}z^{n-k-1}(1-z)^k.$$

Thus

$$a=z^nf(z)a+\bar{z}^nf(\bar{z})a\in R,$$

and so

$$\Lambda_+R\cap\Lambda_-R\subset R.$$

The reverse inclusion is evident.

(8) Suppose that $X\subset A$ is a Λ_+-submodule. There exists a polynomial $p(t)\in\Lambda_+$ with $a=(1-t)p(t)a$ for any $a\in A$, so the multiplication by $1-t$ is an isomorphism $X\to X$. Since $t=-(1-t)\cdot\bar{z}$, we conclude that $t^kX=\bar{z}^kX$, $k\in\mathbf{Z}$.

Similarly, if $Y\subset A$ is a Λ_--submodule, then $t^lY=z^{-l}Y$, $l\in\mathbf{Z}$.

From these two statements it follows that

$$t^k\Lambda_+R=\bar{z}^k\Lambda_+R=\bar{z}^k\bigcup_{n=0}^{\infty}z^{-n}R=\bigcup_{n=0}^{\infty}z^{-n}(\bar{z}^kz^{-l})R=\Lambda_+(\bar{z}^kz^{-l}R).$$

Similarly,

$$t^l\Lambda_-R=\Lambda_-(\bar{z}^kz^{-l}R),$$

and now statement (8) follows from (7).

4.2. Remark. The arguments used in the proof of statement (1), also prove that any module of type K contains at least one P-lattice.

4.3. Let us consider a module A of type K. A $\mathbf{Z}$-homomorphism $f: A\to\mathbf{Q}$ will be called *proper* if it assumes integral values on some P-lattice $R\subset A$. The set of all proper homomorphisms $f: A\to\mathbf{Q}$ will be denoted by $D(A)$.

We will consider $D(A)$ as Λ-module, where for $f\in D(A)$ the homomorphism $(tf)\in D(A)$ is defined by the formula

$$(tf)(a)=f(t^{-1}a),\qquad a\in A.$$

If f assumes integral values on $R\subset A$, then (tf) assumes integral values on $tR\subset A$, which is also a P-lattice.

4.4. Proposition. (1) *For any module A of type K the module $D(A)$ is also of type K.*

(2) *The canonical homomorphism*

$$A \to D(D(A))$$

is an epimorphism with kernel $\mathrm{Tors}_{\mathbf{Z}} A$.

PROOF. (1) Suppose $R \subset A$ is a P-lattice. Any other P-lattice $R_1 \subset A$ contains $(z\bar{z})^n R$ for some $n \geqq 0$ (this follows from 4.1(2)), and so $D(A)$ can be defined as the set of all $\mathbf{Z}$-homomorphisms $f: A \to \mathbf{Q}$, assuming integral values on $(z\bar{z})^n R$ for some $n \geqq 0$.

Let R^* be $\mathrm{Hom}_{\mathbf{Z}}(R; \mathbf{Z})$. Any element of R^* may be uniquely extended (by virtue of 4.1(4)) to an element of $D(A)$. Thus we may consider R^* as a subset of $D(A)$.

Formulas

$$((1-t)f)(a) = f((1-t^{-1})a),$$

$$((1-t^{-1})f)(a) = f((1-t)a), \qquad a \in A,$$

show that the multiplication by $1 - t \in \Lambda$ is an automorphism of $D(A)$. So $D(A)$ can be considered as L-module. In fact,

$$(zf)(a) = f(\bar{z}a), \qquad (\bar{z}f)(a) = f(za)$$

for $a \in A$. Thus, from the above remarks it follows that for any $f \in D(A)$ there exists $n \geqq 0$ with $(z\bar{z})^n f \in R^*$, so $f = (z\bar{z})^{-n} f_1$ for some $f_1 \in R^*$. Since R^* is finitely generated over $\mathbf{Z}$, we conclude that $D(A)$ is finitely generated over Λ. Hence, $D(A)$ is a module of type K and R^* is a P-lattice.

(2) It is clear that $\mathrm{Tors}_{\mathbf{Z}} A$ is contained in the kernel of $A \to D(D(A))$. If $a \in A$ is an element of infinite order, then for some $N > 0$, Na belongs to R and so there exists $f \in R^*$ with $f(a) \neq 0$; thus the kernel of $A \to D(D(A))$ is $\mathrm{Tors}_{\mathbf{Z}} A$. From the commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & R^{**} \\ \cap\downarrow & & \downarrow\cap \\ A & \longrightarrow & D(D(A)) \end{array}$$

we see that, given $y \in D(D(A))$, we can find $n \geqq 0$ with $(z\bar{z})^n y \in R^{**}$ and then realize $(z\bar{z})^n y$ by some $r \in R$. Thus, $(z\bar{z})^{-n} r \in A$ is mapped onto y, and so $A \to D(D(A))$ is onto.

4.5. Let A and B be modules of type K. A *duality pairing* is a $\mathbf{Z}$-bilinear map

$$[\ ,\ ]: A \times B \to \mathbf{Q}$$

satisfying:

(1) $[ta, tb] = [a, b]$ for all $a \in A$, $b \in B$;

(2) the associated map $A \to \mathrm{Hom}_{\mathbf{Z}}(B; \mathbf{Q})$ has kernel $\mathrm{Tors}_{\mathbf{Z}} A$ and image $D(B)$.

Thus, the duality pairing defines a Λ-epimorphism

$$A \to D(B)$$

with kernel $\mathrm{Tors}_{\mathbf{Z}} A$. From Proposition 4.4 it follows that the other associated map

$$B \to \mathrm{Hom}_{\mathbf{Z}}(A; Q)$$

is onto $D(A)$ with kernel $\mathrm{Tors}_{\mathbf{Z}} B$.

4.6. Proposition. *Let* $[\ ,\ ]: A \times B \to \mathbf{Q}$ *be a duality pairing. Then*

(1) *for any P-lattice* $R \subset A$ *the set*

$$R^{\#} = \{b \in B;\ [r, b] \in \mathbf{Z} \textit{ for any } r \in R\}$$

is a P-lattice in B;

(2) $R^{\#\#} = R$;

(3) *if* $R_1, R_2 \subset A$ *are two P-lattices, then* $\Lambda_+ R_1 = \Lambda_+ R_2$ *if and only if* $\Lambda_-(R_1^{\#}) = \Lambda_-(R_2^{\#})$; *similarly,* $\Lambda_- R_1 = \Lambda_- R_2$ *if and only if* $\Lambda_+(R_1^{\#}) = \Lambda_+(R_2^{\#})$.

Proof. (1) $R^{\#}$ is the preimage of R^* (introduced in the proof of Proposition 4.4) under the map

$$B \to D(A).$$

Statement (1) now follows from Proposition 4.4 and the fact that $R^* \subset D(A)$ is a P-lattice.

(2) may be proved similarly.

(3) According to Proposition 4.1(5), the equality $\Lambda_+ R_1 = \Lambda_+ R_2$ takes place if and only if $z^n R_1 \subset R_2$ and $z^m R_2 \subset R_1$ for some $n \geqq 0$, $m \geqq 0$. But in general,

$$\bar{z}^k R^{\#} = (z^{-k} R)^{\#}, \qquad z^k R^{\#} = (\bar{z}^{-k} R)^{\#},$$

where R is a P-lattice, $R \subset A$, and k is an integer. Thus, the inclusion $z^n R_1 \subset R_2$ gives

$$R_2^{\#} \subset (z^n R_1)^{\#} = \bar{z}^{-n} R_1^{\#}, \qquad \bar{z}^n R_2^{\#} \subset R_1^{\#}$$

and similarly

$$\bar{z}^m R_1^\# \subset R_2^\#.$$

Applying 4.1(6) we get

$$\Lambda_-(R_1^\#) = \Lambda_-(R_2^\#).$$

The other statement can be proved similarly.

4.7. PROPOSITION. *Suppose A is a module of type K and*

$$[\ ,\ ]: A \times A \to \mathbf{Q}$$

is an ε-symmetric duality pairing, $\varepsilon = \pm 1$. Then A contains a self-dual P-lattice, $R = R^\#$.

PROOF. Let $R \subset A$ be an arbitrary P-lattice. By virtue of the identity

$$((z\bar{z})^{-n}R)^\# = (z\bar{z})^n R^\#,$$

for sufficiently large n we will have

$$((z\bar{z})^{-n}R)^\# \subset R \subset (z\bar{z})^{-n}R.$$

Thus, we are able to assume that

$$R^\# \subset R$$

(if this is not true take $(z\bar{z})^{-n}R$ instead of R).

Assuming $R^\# \subset R$, consider the set

$$S = \{r \in R; z^n r \in R^\# \text{ for some } n \geqq 0\}.$$

S is a P-lattice in S. We will show that S is self-dual.

Suppose $r_1, r_2 \in S$ and $z^n r_1, z^n r_2 \in R^\#$. There exists an integral polynomial $f(z)$ with the property

$$1 = z^n f(z) + \bar{z}^n f(\bar{z})$$

(constructed in the proof of statement 4.1(7)). Now,

$$r_\nu = z^n f(z) r_\nu + \bar{z}^n f(\bar{z}) r_\nu, \qquad \nu = 1, 2$$

and so

$$[r_1, r_2] = [z^n f(z) r_1, z^n f(z) r_2] + [z^n f(z) r_1, \bar{z}^n f(\bar{z}) r_2]$$
$$+ [\bar{z}^n f(\bar{z}) r_1, z^n f(z) r_2] + [\bar{z}^n f(\bar{z}) r_1, \bar{z}^n f(\bar{z}) r_2].$$

Since $z^n r_\nu \in R^\#$, the three first summands are integral. But the fourth summand is equal to $[z^n f(z) r_1, z^n f(z) r_2]$ and so it is also integral. This proves $S^\# \subset S$.

For some $n > 0$ the lattice $(z\bar{z})^n R$ is contained in $R^\#$. Thus, for $r \in R$, $\bar{z}^n r \in S$. If $r_1 \in S^\#$, then $[r_1, \bar{z}^n r] = [z^n r_1, r]$ is an integer for any $r \in R$, and so $z^n r_1 \in R^\#$ and $r_1 \in S$. This proves $S^\# \subset S$.

4.8. Definition. Let R be a P-module. A submodule $T \subset R$ is *basic* if

(a) T contains $(z\bar{z})^k R$ for some $k \geqq 0$;

(b) if $r \in R$ and $Nr \in T$ for some $N \neq 0$, then $r \in T$;

(c) the kernel of the homomorphism $T \to T$ given by multiplication by $z\bar{z} \in P$ is contained in $\mathrm{Tors}_{\mathbf{Z}}\, T$.

It was shown in [F3], 6.2, that every P-module finitely generated over $\mathbf{Z}$ has a unique basic submodule.

Assume R is a P-module finitely generated over $\mathbf{Z}$ with no $\mathbf{Z}$-torsion. Let T be its basic submodule and let $M \subset R$ be the set

$$\{r \in R;\, (z\bar{z})^k r = 0 \text{ for some } k \geqq 0\}.$$

Then it is clear that

$$R = T \oplus M.$$

In the general case, if we would not have assumed that R has no $\mathbf{Z}$-torsion, then $R = T + M$ and $T \cap M$ is equal to the torsion subgroup of R.

4.9. Now we are going to apply the algebraic notions of this section to the geometry.

Let $K = (S^{n+2}, k^n)$ be a knot, $X = S^{n+2} - k$ its complement and $p: \tilde{X} \to X$ the infinite cyclic cover. Any Seifert manifold $V \subset S^{n+2}$ of K admits a lifting $f: \mathrm{int}\, V \to \tilde{X}$ and the image of the induced map

$$f_*: H_i V \to H_i \tilde{X} = A_i(K)$$

is a P-lattice, which will be denoted by $P_i(f)$.

On the other hand, a Seifert manifold has an obvious framing ν, and the lifting $f: \mathrm{int}\, V \to \tilde{X}$ defines also a lifting

$$\mu: X(V, \nu) \to \tilde{X},$$

the manifold $X(V, \nu)$ having been constructed in 1.1. The image of the induced map

$$\mu_*: H_i(X(V, v)) \to \tilde{X}$$

is a Λ_+-lattice in $\tilde{X}$, which will be denoted by $\Lambda_{+i}(f)$.

4.10. PROPOSITION. *There exists a bilinear map (the Milnor pairing)*

$$[\ ,\ ]: H_i(\tilde{X}) \times H_{n+1-i}(\tilde{X}) \to \mathbf{Q},$$

having the following properties:

(1) *if $a \in H_i(V)$ and $b \in H_{n+1-i}(V)$ belong to the corresponding basic submodules of $H_i V$ and $H_{n+1-i}(V)$, then*

$$[f_*(a), f(_*b)] = \langle a, b \rangle,$$

where $\langle\ ,\ \rangle$ denotes the intersection number;

(2) $[\ ,\ ]$ *is a duality pairing in the sense of* 4.5;

(3) *P-lattices $P_i(f)$ and $P_{n+1-i}(f)$ are dual to each other*;

(4) *the lattice $\Lambda_{+i}(f)$ coincides with the Λ_+-lattice, generated by $P_i(f)$.*

PROOF. First we will construct $[\ ,\ ]$; basically, property (1) will be taken as the definition.

Let $x \in H_i\tilde{X}$, $y \in H_{n+1-i}\tilde{X}$. According to Lemma 6.3 of [F3], there exist elements $a \in H_i V$ and $b \in H_{n+1-i}(V)$, belonging to the basic submodules, and integers $N \neq 0$, $M \neq 0$, such that

$$Nx = f_*(a), \qquad My = f_*(b).$$

Then we define

$$[x, y] = \frac{1}{NM} \langle a, b \rangle \in \mathbf{Q}.$$

The correctness of this definition follows from Lemma 6.3 of [F3].

Property (1) is obviously satisfied.

To prove (2), note that Proposition 1.2 of [F3] implies

$$[ta, tb] = [a, b]$$

for all $a, b \in H_*\tilde{X}$. Consider the associated map

$$H_i\tilde{X} \xrightarrow{\text{ass}} \mathrm{Hom}_{\mathbf{Z}}(H_{n+1-i}(\tilde{X}); \mathbf{Q});$$

we have the following commutative diagram:

$$\begin{array}{ccc}
\beta H_i V & \xrightarrow{\text{ass}_1} & \operatorname{Hom}(\beta H_{n+1-i}(V); \mathbf{Z}) \\
\downarrow \cap & & \\
H_i V & & \downarrow \varphi \\
f_* \downarrow & & \\
H_i \tilde{X} & \xrightarrow{\text{ass}} & \operatorname{Hom}(H_{n+1-i}(\tilde{X}); \mathbf{Q}).
\end{array}$$

Here β means the operation of taking the basic submodule, ass_1 is the map associated with the pairing

$$\langle\ ,\ \rangle_1 : \beta H_i V \times \beta H_{n+1-i}(V) \to \mathbf{Z}$$

which is the restriction of the intersection form $\langle\ ,\ \rangle$, and the map φ acts as follows: given $g : \beta H_{n+1-i}(V) \to \mathbf{Z}$, consider $\beta H_{n+1-i} V$ as embedded in $H_{n+1-i}(\tilde{X})$, then Lemma 6.3 of [F3] says that g admits a unique extension $\tilde{g} : H_{n+1-i}(\tilde{X}) \to \mathbf{Q}$ and we set $\tilde{g} = \varphi(g)$. Commutativity of this diagram follows from the definitions. The vertical map on the right is a monomorphism, the vertical map on the left has the torsion subgroup as its kernel (see 6.3 of [F3]). Thus, the kernel of ass coincides with the torsion subgroup.

It is clear that the image of ass lies in the set of proper homomorphisms, $D(H_{n+1-i}(\tilde{X}))$. To prove that the image of ass coincides with $D(H_{n+1-i}(\tilde{X}))$, it is enough to show that

$$\langle\ ,\ \rangle_1 : \beta H_i V \times \beta H_{n+1-i}(V) \to \mathbf{Z}$$

induces an epimorphism

$$\beta H_i V \to \operatorname{Hom}(\beta H_{n+1-i}(V); \mathbf{Z}).$$

To do this, suppose we are given $g : \beta H_{n+1-i}(V) \to \mathbf{Z}$. Since $\beta H_{n+1-i}(V)$ is a pure subgroup, g may be extended to a homomorphism

$$\bar{g} : H_{n+1-i}(V) \to \mathbf{Z}.$$

We shall consider the unique extension $\bar{g}$ satisfying $\bar{g}(b) = 0$ for all $b \in \tau H_{n+1-i}(V)$, where

$$\tau H_{n+1-i}(V) = \{b \in H_{n+1-i}(V);\ (z\bar{z})^n b = 0 \text{ for some } n \geqq 0\}.$$

By virtue of the Poincaré duality, there exists $a \in H_i V$ with $\langle a, b\rangle = \bar{g}(b)$

for all $b \in H_{n+1-i}(V)$. Write $a = a_1 + a_2$, where $a_1 \in \beta H_i(V)$, $a_2 \in \tau H_i(V)$. From the equalities

$$\langle a_1, \tau H_{n+1-i}(V)\rangle = 0, \qquad \langle a_2, \beta H_{n+1-i}(V)\rangle = 0$$

it follows that

$$\langle a_1, b\rangle = g(b)$$

for all $b \in H_{n+1-i}(V)$ and, thus,

$$\mathrm{ass}_1(a_1) = g.$$

This completes the proof of (2).

(3) follows from the arguments described immediately above and from the remark that

$$P_i(f) = \mathrm{im}[f_*: \beta H_i(V) \to H_i\tilde{X}].$$

(4) might be proved by the standard arguments used in the proof of Proposition 2.2 in [F3].

4.11. Proof of Theorem 2.3. Suppose that all conditions of the theorem are satisfied. Lattices $S_r \subset H_r\tilde{X}$ are given just for $r \leqq q = [(n+1)/2]$. Define $S_r \subset H_r\tilde{X}$ for $r > q$ to be $(S_{n+1-r})^{\#}$ — the lattice dual to S_{n+1-r} relative to the Milnor form. Let $C_r = \Lambda_+ S_r$, $r = 1, 2, \ldots, n$; in other words, C_r is the Λ_+-lattice, generated by S_r.

The realization Theorem 1.5, after having been applied to the manifold (S^{n+2} — small tubular neighbourhood of the knot k), gives a Seifert manifold $V^{n+1} \subset S^{n+2}$ of K with the properties:

(a) V is simply connected;

(b) it is minimal in the sense of 2.5;

(c) for some lifting f: int $V \to \tilde{X}$, where $\tilde{X}$ is the infinite cyclic covering of the complement $X = S^{n+2} - k$, we have $\Lambda_{+,r}(f) = t^{\alpha_r} C_r$, $r = 1, 2, \ldots, n$, α_r is an integer. Here we use notations introduced in 4.9.

Let D_r denote $P_r(f)$. D_r is a P-lattice in $H_r\tilde{X}$ and $\Lambda_+ D_r = t^{\alpha_r} C_r$ (because of (c) and statement (4) of 4.10). From part (4) of 4.10 we know that

$$D_r = (D_{n+1-r})^{\#},$$

and thus for any $r = 1, 2, \ldots, q$ we have

$$\Lambda_+ D_r = t^{\alpha_r}\Lambda_+ S_r,$$

$$\Lambda_+ D_r^\# = t^{\alpha_{n+1-r}}\Lambda_+ S_r^\# .$$

If we put $a = \alpha_r$, $b = \alpha_{n+1-r}$, then the second equality can be rewritten as

$$\Lambda_+ D_r^\# = \Lambda_+(t^b S_r^\#) = \Lambda_+(t^b S_r)^\#.$$

Using 4.6(3) we get

$$\Lambda_- D_r = \Lambda_-(t^b S_r) = t^b \Lambda_- S^r$$

and

$$\begin{aligned} D_r &= \Lambda_+ D_r \cap \Lambda_- D_r \\ &= t^a \Lambda_+ S_r \cap t^b \Lambda_- S_r \\ &= \bar{z}^a z^{-b} S_r = \bar{z}^{\alpha_r} z^{-\alpha_{n+1-r}} S_r, \end{aligned}$$

where we have used 4.1(7) and 4.1(8).

Thus, we have proved that the map

$$\bar{z}^{-\alpha_r} z^{\alpha_{n+1-r}} f_* : H_r V \to H_r \tilde{X}$$

provides a monomorphism with image S_r. If n is odd, $n = 2q - 1$, and $r = q$, then this homomorphism is equal to $(-t)^{-\alpha_q} f_*$ which clearly takes the Milnor form to the intersection form of V (by virtue of 4.10(1)).

This completes the proof.

§5. Proofs of Lemmas 1.6, 1.7 and 1.8

The proofs of Lemmas 1.6 and 1.7 will in turn use Lemmas 5.1 and 5.2, which are stated below.

In Lemmas 5.1 and 5.2 it is assumed that a smooth, compact, n-dimensional manifold Y^n is given and two submanifolds, V_0, $V_1 \subset \partial Y$ are distinguished. It is also supposed that $n \geqq 6$, the manifolds V_0, V_1, Y are simply connected and the cobordism on the boundary $\delta Y = \mathrm{cl}(\partial Y - (V_0 \cup V_1))$ between ∂V_0 and ∂V_1 is trivial (Fig. 1).

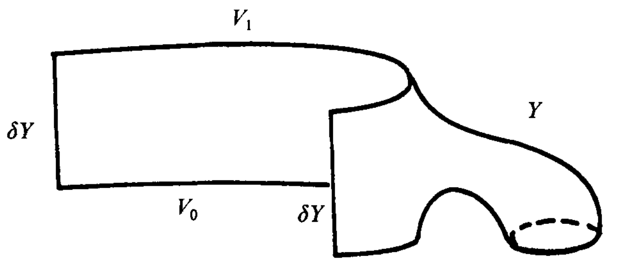

Fig. 1.

5.1. Lemma. *For any $k \leqq n-3$ there exists a smooth simply connected $(n-1)$-dimensional submanifold $W \subset Y$ with $\partial W = W \cap \partial Y = \partial V_0$ such that $Y - W$ consists of two components, and for the component N, containing V_1 (cf. Fig. 2), the following is true*:

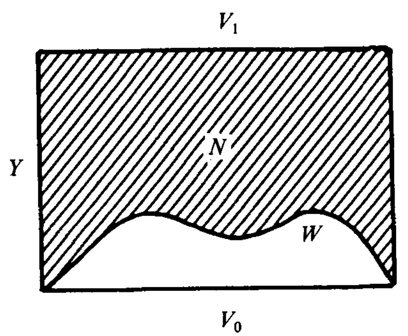

Fig. 2.

(1) *the induced by inclusion homomorphism $H_i(N, V_1) \to H_i(Y, V_1)$ is an isomorphism for all $i \leqq k-1$ and is an epimorphism for $i = k+1$;*

(2) *for $i = k$ and for $i > k+1$ the group $H_i(N, V_1)$ is trivial.*

5.2. Lemma. *For any $k \leqq n-3$ and for any class $z \in H_k(Y, V_1)$ there exists a smooth simply connected $(n-1)$-dimensional submanifold $W \subset Y$ with $\partial W = W \cap \partial Y = \partial V_0$, such that $Y - W$ consists of two components and for the component N, containing V_1, the following is true*:

(1) *the homomorphism induced by inclusion* $H_i(N, V_1) \to H_i(Y, V_1)$ *is an isomorphism for* $i \leqq k-1$;

(2) *the group* $H_k(N, V_1)$ *is isomorphic to* Z *and the class* z *is the image of its generator under the homomorphism* $H_k(N, V_1) \to H_k(Y, V_1)$;

(3) *the group* $H_i(N, V_1)$ *is trivial for* $i > k$.

5.3. In the proofs of Lemmas 5.1 and 5.2 we shall use the following well known fact: if an $(n-1)$-dimensional manifold W is obtained from another $(n-1)$-dimensional V by a spherical modification of index i, where $2 \leqq i \leqq n-3$, then W is also simply connected.

5.4. Proof of Lemma 5.1. According to Smale's theorem ([M1], th. 6.1) there exists a Morse function $f: Y \to [0, 1]$, which is equal to 0 on V_1 and 1 on V_0 and such that for each j, $0 \leqq j \leqq n$, the number of critical points of f of index j is equal to $b_j + q_j + q_{j-1}$, where b_j is the rank of the group $H_j(Y, V_1)$, and q_j is the minimal number of generators of its torsion subgroup. Moreover, the restriction of f on the boundary cobordism $\delta Y = \mathrm{cl}(\partial Y - (V_0 \cup V_1))$ has no critical points. The function f gives rise to a handle decomposition of Y, with the handles glued in the order of indices to a collar of V_1 in Y. It is clear that $b_1 = 0$, $q_1 = 0$. By the Poincaré duality,

$$H_{n-1}(Y, V_1) \approx H^1(Y, V_0)$$

and

$$H_{n-2}(Y, V_1) \approx H^2(Y, V_0) \approx \mathrm{Hom}(H_2(Y, V_0); Z).$$

These imply that $b_n = b_{n-1} = 0$ and $q_n = q_{n-1} = q_{n-2} = 0$ and it follows that the function f has no critical points of indices $0, 1, n-1, n$.

Let Y' be obtained as the result of gluing of all handles of indices $\leqq k-1$. We may suppose that δY is contained in Y'.

Let $2 \leqq j \leqq n-2$. In the group C_j of the chain complex, generated by f, we may choose the following base,

$$z_1^j, z_2^j, \ldots, z_\mu^j, \beta_1^j, \ldots, \beta_q^j,$$

where $\mu = b_j + q_j$, $q = q_{j-1}$, the elements z_i^j form a base in the group of j-dimensional cycles Z_j, and the boundaries of the elements β_i^j form a base in the group B_{j-1} of $(j-1)$-dimensional boundaries. By Theorem 7.6 of [M1], we may assume that this base realizes the j-dimensional handles of Y. We shall denote by H_i^j the handle realized by z_i^j and denote by h_i^j the handle realizing β_i^j.

Let us suppose that the construction of the previous paragraph has been performed for $j = k$ and $j = k + 1$.

Since the handle H_i^{k+1} realizes a cycle, the intersection numbers of its attaching k-dimensional α-sphere with the β-spheres of handles of index k are all equal to zero. Thus using the Whitney lemma ([M], th. 6.6), we may isotope handles H_i^{k+1} such that they do not intersect handles of index k.

Assuming handles H_i^{k+1} do not intersect handles of index k, consider the submanifold $N \subset Y$, which is the union

$$Y' \cup h_1^k \cup h_2^k \cup \cdots \cup h_{q_{k-1}}^k \cup H_1^{k+1} \cup \cdots \cup H_{\mu_{k+1}}^{k+1}.$$

It is clear that the homomorphism $H_j(N, V_1) \to H_j(Y, V_1)$ induced by the inclusion is an isomorphism for $j \leqq k-1$ and an epimorphism for $j = k+1$. Besides, $H_j(N, V_1) = 0$ for $j = k$ and $j > k+1$.

The boundary of the constructed N satisfies $\partial N \cap \partial Y = \partial Y - \text{int } V_0$. Let us denote $W = \text{cl}(\partial N \cap \text{int } Y)$.

The lemma will follow if we prove that W is simply connected. For $k + 1 \leqq n - 3$ it is a consequence of the remark 5.3. If $k + 1 = n - 2$, then, using the dual handle decomposition, we may note that W is obtained from V_0 by a surgery of indices 2 and 3. And now we may use the remark 5.3 once more, due to the assumptions: $n \geqq 6$ and V_0 is simply connected.

The lemma follows.

5.5. Proof of Lemma 5.2. Let us consider the exact Morse function $f: Y \to [0, 1]$ and the induced handle decomposition on handles of the form H_i^k and h_i^k similar to the proof of Lemma 5.1.

Let $Y' \subset Y$ be the union of all handles of indices $\leqq k-1$ and also of the handles $h_1^k, h_2^k, \ldots, h_{q_{k-1}}^k$. We shall suppose that Y' contains $\partial Y - \text{int } V_0$. It is clear that $H_j(Y', V_1) = 0$ for $j \geqq k$ and the inclusion $(Y', V_1) \to (Y, V_1)$ induces an isomorphism $H_j(Y', V_1) \to H_j(Y, V_1)$ for $j \leqq k-1$. Let $Q = Y - \text{int } Y'$ and $U = Q \cap Y'$, cf. Fig. 3.

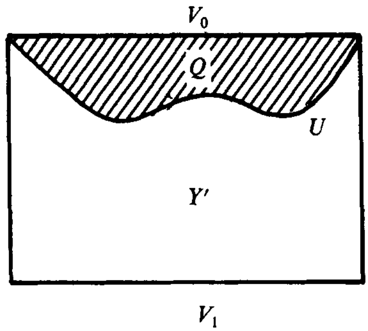

Fig. 3.

For all i we have $H_i(Q, U) = H_i(Y, Y')$ and from the exact sequence

$$\cdots \to H_i(Y', V_1) \to H_i(Y, V_1) \to H_i(Y, Y') \to \cdots$$

we find: $H_i(Y, Y') = 0$ for $i \leqq k - 1$ and the inclusion induces an isomorphism $H_k(Y, V_1) \approx H_k(Y, Y')$.

Thus, $H_i(Q, U) = 0$ for $i \leqq k - 1$ and the group $H_k(Q, U)$ is naturally isomorphic to $H_k(Y, V_1)$. Let $z' \in H_k(Q, U)$ be the image of z under this isomorphism. The pair (Q, U) is $(k - 1)$-connected (by virtue of remark 5.3) and applying Corollary 1.1 from [H2] we get an embeddeding $D^k \to Q$ with $D^k \cap \partial Q = D^k \cap U = S^{k-1}$, realizing z'. Let N be the union of Y' and a tubular neighbourhood of D^k in Q.

Consider the following diagram, where all homomorphisms are induced by inclusions:

$$\begin{array}{ccc} H_j(N, V_1) & \xrightarrow{\beta} & H_n(Y, V_1) \\ & {\scriptstyle\alpha}\nwarrow \quad \nearrow{\scriptstyle\gamma} & \\ & H_j(Y', V_1) & \end{array}$$

If $j \leqq k - 1$ then γ is an isomorphism. On the other hand, α is an isomorphism for $j \leqq k - 2$ since $H_j(N, Y') \approx H_j(D^k, \partial D^k)$. Considering the following diagram:

$$\begin{array}{ccccccccc}
0 \to H_k(Y, V_1) & \xrightarrow{\sigma} & H_k(Y, Y') & \xrightarrow{\partial} & H_{k-1}(Y', V_1) & \xrightarrow{\gamma} & H_{k-1}(Y, V_1) \to 0 \\
\uparrow \beta & & \uparrow \nu & & \uparrow = & & \uparrow \beta \\
0 \to H_k(N, V_1) & \xrightarrow{\kappa} & H_k(N, Y') & \to & H_{k-1}(Y', V_1) & \xrightarrow{\alpha} & H_{k-1}(N, V_1) \to 0
\end{array}$$

one may deduce that α is an isomorphism for $j = k - 1$ as well.

Thus, we have proved that for $j \leqq k - 1$ both α and γ are isomorphisms. So for these values of j, β is an isomorphism too.

From the above diagram it follows that the homomorphisms $\sigma: H_k(Y, V_1) \to H_k(Y, Y')$ and $\kappa: H_k(N, V_1) \to H_k(N, Y')$ are isomorphisms. Since the group $H_k(N, Y')$ is evidently isomorphic to Z, then also $H_k(N, V') \approx Z$. Since the image of z under σ coincides with the image of a generator of $H_k(N, Y')$ under ν, the homomorphism $\beta: H_k(N, V_1) \to H_k(Y, V_1)$ maps a generator of $H_k(N, V_1)$ into z.

We now have only to denote $W = \mathrm{cl}(\partial N \cap \mathrm{int}\, Y)$; W is simply connected (by arguments similar to those in the proof of Lemma 5.1).

The lemma is proved.

5.6. Proof of Lemma 1.6. Let $2 \leqq k \leqq n - 3$ and let (V^{n-1}, ν) be a simply connected, proper, frame submanifold in M^n, satisfying conditions (I), (II), (III_i), (IV_i) of Theorem 1.5 for all $i < k$. Let the cobordism $(Y; V_0, V_1)$ be obtained by cutting of M along V and let $\psi: Y \to M$ be the natural map (cf. 1.1). Apply Lemma 5.1 to the cobordism $(Y; V_0, V_1)$ and to the number k. The submanifold $W^{n-1} \subset Y$ which is given by this lemma has a framing ω, with the vectors of ω directed inside the component N of the complement $Y - W$ con- taining V_1. The image of (W^{n-1}, ω) under ψ is a frame submanifold of M, satisfying conditions (I) and (II) of Theorem 1.5; we shall identify it with (W^{n-1}, ω) and denote it by the same symbol.

Suppose that the manifolds $X_V = X(V^{n-1}, \nu)$ and $X_W = X(W^{n-1}, \omega)$ are constructed as explained in 1.1. There are natural inclusions $tX_V \subset X_W \subset X_V$, and the induced homomorphism $H_r(X_W, tX_V) \to H_r(X_V, tX_V)$ is an isomorphism for $r \leqq k - 1$ and is an epimorphism for $r = k + 1$, the group $H_r(X_W, tX_V)$ being trivial for $r = k$ and for $r \geqq k + 1$. This follows from the commutative diagram

$$\begin{array}{ccc} H_r(X_W, tX_V) & \longrightarrow & H_r(X_V, tX_V) \\ \downarrow \approx & & \downarrow \approx \\ H_r(N, V_1) & \longrightarrow & H_r(Y, V_1) \end{array}$$

whose vertical isomorphisms are excision isomorphisms and whose lower horizontal homomorphism satisfies conditions (1) and (2) of Lemma 5.1.

We shall later show that (a) the homomorphism $j_*: H_r(X_W) \to H_r(X_V)$ is an isomorphism for $r \leqq k-1$ and (b) the homomorphism $i_*: H_k(tX_V) \to H_k(X_W)$ is an epimorphism and its kernel coincides with the kernel of the homomorphism $H_k(tX_V) \to H_k(X_V)$, induced by the inclusion. Let us complete the proof of the lemma supposing (a) and (b) true. By virtue of (a), the homomorphism $\mu_*: A_r(W^{n-1}, \omega) \to H_r(M_\xi)$ is a monomorphism and its image coincides with $B_r(V^{n-1}, \nu)$ for $r < k$. In other words, (W^{n-1}, ω) satisfies conditions (III_r) and (IV_r) for $r < k$. By virtue of (b) $B_k(W^{n-1}, \omega) = tB_k(V^{n-1}, \nu)$ and now we shall show that the kernel P of the homomorphism $\mu_*: A_k(W^{n-1}, \omega) \to H_k(M_\xi)$ is in some sense smaller than the kernel Q of the homomorphism $\mu_*: A_k(V^{n-1}, \nu) \to H_k(M_\xi)$. In fact,

$$P = \{a \in A_k(W^{n-1}, \omega);\ \exists\, m \geqq 0,\ t^m a = 0\},$$

$$Q = \{a \in A_k(V^{n-1}, \nu);\ \exists\, m \geqq 0,\ t^m a = 0\},$$

and so the restriction of i_* maps Q onto P and the kernel of the restriction $i_* |_Q$ coincides with $\{q \in Q;\ tq = 0\}$. Thus, P is isomorphic to

$$Q/\{q \in Q;\ tq = 0\}$$

and now it is clear in which sense P is less that Q (note that $A_k(V^{n-1}, \nu)$ and $A_k(W^{n-1}, \omega)$ are finitely generated over Λ_+ and so P and Q are finitely generated over Z).

Let us iterate the construction, which we have applied to V in order to obtain W. As a result we shall get a sequence (W_s^{n-1}, ω_s), where $s = 1, 2, \ldots$ of framed submanifolds with $(W_1, \omega_1) = (W^{n-1}, \omega)$, satisfying (I), (II), (III_r), (IV_r) for $r \leqq k-1$ and $B_k(W_s^{n-1}, \omega_s) = t^s B_k(V^{n-1}, \nu)$. Besides, the kernel of the homomorphism

$$\mu_*: A_k(W_s^{n-1}, \omega_s) \to H_k(M_\xi)$$

is isomorphic to

$$Q/\{q \in Q;\ t^s q = 0\}.$$

It is clear that for sufficiently large s this group is trivial and so condition (III_k) is satisfied.

To complete the proof we have only to prove statements (a) and (b). From the exact homology sequence of the triple (X_V, X_W, tX_V), using the fact that $H_r(X_W, tX_V) \to H_r(X_V, tX_V)$ is an isomorphism for $r \leqq k-1$, we obtain $H_r(X_V, X_W) = 0$ for $r \leqq k-1$ and from this it follows that $j_*: H_r(X_W) \to H_r(X_V)$ is an isomorphism for $r \leqq k-2$. To show that it is also true for $r = k-1$ consider the following commutative diagram with exact columns and rows:

$$
\begin{array}{ccccccc}
H_{k-1}(X_V, tX_V) & & & & & & \\
\approx\uparrow\varphi_1 & & & & & & \\
H_{k-1}(X_W, tX_V) & & & & & & \\
\uparrow & & & & & & \\
H_k(X_V, X_W) & \xrightarrow{\varphi_4} & H_{k-1}(X_W) & \xrightarrow{j_*} & H_{k-1}(X_V) & \longrightarrow & 0 \\
\approx\uparrow\varphi_2 & & \uparrow & & \uparrow = & & \\
H_k(X_V, tX_V) & \xrightarrow{\varphi_3} & H_{k-1}(tX_V) & \xrightarrow{\varphi_5} & H_k(X_V) & & \\
\uparrow & & & & & & \\
0 & & & & & &
\end{array}
$$

The homomorphism ϕ_1 is an isomorphism (see the beginning of the proof), and so ϕ_2 is also an isomorphism. By virtue of the assumptions of the lemma, (III_{k-1}) is satisfied and so ϕ_5 is a monomorphism, and so $\phi_3 = 0$ and also $\phi_4 = 0$. This proves that j_* is mono.

j_* is also onto; this follows from the equality $H_{k-1}(X_V, X_W) = 0$ which was obtained above.

To prove (b) consider the diagram

$$
\begin{array}{ccccccc}
H_{k+1}(X_W, tX_V) & \xrightarrow{\psi_4} & H_k(tX_V) & \xrightarrow{i_*} & H_k(X_W) & \longrightarrow & 0 \\
\downarrow\psi_3 & & \downarrow = & & \downarrow & & \\
H_{k+1}(X_V, tX_V) & \xrightarrow{\psi_2} & H_k(tX_V) & \xrightarrow{\psi_1} & H_k(X_V) & \longrightarrow & 0
\end{array}
$$

Due to the fact that ψ_3 is an epimorphism (see the beginning of the proof), we find

$$\ker(\psi_1) = \operatorname{im}(\psi_2) = \operatorname{im}(\psi_2 \circ \psi_3) = \operatorname{im}(\psi_4) = \ker(i_*),$$

which proves (b) and the lemma follows.

5.7. Proof of Lemma 1.7. Let $2 \leqq k \leqq n-3$ and $(V^{n-1}, v) \subset M^n$ be a simply connected proper framed submanifold, satisfying (in the notation of Theorem 1.5) conditions (I), (II), (III_r) for $r \leqq k$ and (IV_r) for $r \leqq k-1$. Suppose also that $tB_k(V, \nu) \subset C_k \subset B_k(V, \nu)$ and the factor group $C_k/tB_k(V, \nu)$ is cyclic, generated by some $b \in B_k(V, \nu)/tB_k(V, \nu)$. Let the cobordism $(Y; V_0, V_1)$ be obtained by cutting M along V and $\psi : y \to M$ be the natural map (see 1.1). Due to conditions (III_{k-1}) and (III_k), the group $B_k(V, \nu)/tB_k(V, \nu)$ is isomorphic to $H_k(X_V, tX_V)$; let $z \in H_k(X_V, tX_V)$ denote the image of b under the evident isomorphism (here $X_V = X(V, \nu)$ is the manifold of the subsection 1.1). By the excision axiom $H_k(X_V, tX_V) \approx H_k(Y, V_1)$ and let $z' \in H_k(Y, V_1)$ be the image of z.

Apply Lemma 5.2 to the cobordism $(Y; V_0, V_1)$, the number k and the class z'. The submanifold $W^{n-1} \subset y$, which is given by this lemma, has a framing ω, whose vectors are directed into the interior of the component of $Y - W$, containing V_1, which we denote by N. The image of (W, ω) under ψ is a framed submanifold in M, satisfying (I), (II) of Theorem 1.5; we will identify it by (W^{n-1}, ω) and denote it by the same symbol.

Let $X_W = X(W, \omega)$ be built as in 1.1. There are inclusions

$$tX_V \xrightarrow{i} X_W \longrightarrow X_V,$$

and conditions (1), (2), (3) of Lemma 5.2 imply that the homomorphism induced by the inclusion

$$H_r(X_W, tX_V) \to H_r(X_V, tX_V)$$

is an isomorphism for $r \leqq k-1$. Besides, the group $H_r(X_W, tX_V)$ is trivial for $r > k$, and it is $\mathbf{Z}$ for $r = k$; moreover, for $r = k$ the image of a generator of $H_k(X_W, tX_V)$ in $H_k(X_V, tX_V)$ is z.

Let us show that (a) the homomorphism

$$j_* : H_r(X_W) \to H_r(X_V)$$

is an isomorphism for $r \leqq k-1$, and (b) for $r = k$ its image coincides with $C_k \subset H_k(X_V)$.

In fact, similarly to the proof of Lemma 1.6, $H_r(X_V, X_W) = 0$ for $r \leqq k-1$ and so $j_* : H_r(X_W) \to H_r(X_V)$ is an isomorphism for $r \leqq k-2$. To show that it is also true for $r = k-1$, let us consider the following commutative diagram:

$$
\begin{array}{ccccccc}
H_{k-1}(X_V, tX_V) & & & & & & \\
\approx\uparrow \varphi_1 & & & & & & \\
H_{k-1}(X_W, tX_V) & & & & & & \\
\uparrow & & & & & & \\
H_k(X_V, X_W) & \xrightarrow{\varphi_5} & H_{k-1}(X_W) & \xrightarrow{j_*} & H_{k-1}(X_V) & \longrightarrow & 0 \\
\uparrow \varphi_2 & & \uparrow & & \uparrow = & & \\
H_k(X_V, tX_v) & \xrightarrow{\varphi_4} & H_{k-1}(X_V) & \xrightarrow{\varphi_3} & H_{k-1}(X_V) & &
\end{array}
$$

φ_1 is an isomorphism, which implies that φ_2 is an epimorphism. Besides, φ_3 is a monomorphism (due to the condition (III_{k-1})) and so $\varphi_4 = 0$. From these, $\varphi_5 = 0$ follows and so j_* is an isomorphism for $r = k - 1$.

To prove the statement (b) consider the commutative diagram with exact columns and rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_k(tX_V) & \xrightarrow{i_*} & H_k(X_W) & \xrightarrow{\psi_4} & H_k(X_W, tX_V) & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow & & \downarrow \psi_1 & & \\
0 & \longrightarrow & H_k(tX_V) & \xrightarrow{\psi_3} & H_k(X_V) & \xrightarrow{\psi_2} & H_k(X_V, tX_V) & \longrightarrow & 0
\end{array}
$$

According to the construction, $C_k = \psi_2^{-1}(\mathrm{im}\ \psi_1)$. Since ψ_4 is an epimorphism,

$$
\begin{aligned}
\psi_2^{-1}(\mathrm{im}\ \psi_1) &= \psi_2^{-1}(\mathrm{im}(\psi_1 \circ \psi_4)) = \psi_2^{-1}(\mathrm{im}(\psi_2 \circ j_*)) \\
&= \mathrm{im}\, j_* + \mathrm{im}\ \psi_3 = \mathrm{im}\, j_* + \mathrm{im}(j_* \circ i_*) = \mathrm{im}\, j_*.
\end{aligned}
$$

This proves (b).

Arguments similar to those of Lemma 1.6 show that from conditions (a) and (b) it follows that the constructed manifold (W, ω) satisfies all necessary conditions besides (III_k). After application to (W, ω) of the construction of Lemma 1.6, we will receive another framed submanifold $(W_1, \omega_1) \subset M$ which satisfies all the conditions (I), (II), (III_r) and (IV_r) for all $r \leqq k$.

The lemma is proved.

5.8. PROOF OF LEMMA 1.8. Let us assume at first that $B \subset C \subset t^{-1}B$. It is clear that the factor group C/B is finitely generated over Z. Suppose that the classes of the elements $c_1, c_2, \ldots, c_N \in C$ generate C/B. Let us denote $A_0' = B$, $A_i' = B + (c_1, \ldots, c_i)$, where $i = 1, 2, \ldots, N$, and the symbol $(c_1, \ldots, c_i)$ denotes the subgroup generated by elements $c_1, c_2, \ldots, c_i$. It is clear that A_i' is a Λ_+-lattice in H and $A_i' \subset A_{i+1}' \subset t^{-1}A_i'$. Besides, the factor group A_{i+1}'/A_i' is cyclic. Thus, if one puts $A_i = t^i A_i'$ for $i = 0, 1, \ldots, N$, then all conditions of the lemma will be satisfied.

Let us consider now the general case. From the fact that C generates H over Λ and is a Λ_+-submodule, it follows that for any $h \in H$ there is an integer $\alpha \geqq 0$ with $t^\alpha h \in C$. There exists $\alpha_1 \geqq 0$ such that $t^{\alpha_1}B \subset C$. Similarly, there is an integer $\alpha_2 \geqq 0$ with $t^{\alpha_2}C \subset B$. Thus, $t^{\alpha_1}B \subset C \subset t^{-\alpha_2}B$. Denote $C_k = C \cap t^{\alpha_1 - k}B$, where $k = 0, 1, \ldots$. Then $C_0 = t^{\alpha_1}B$ and C_k coincides with C for sufficiently large k. From the relation $C_{k-1} \subset C_k \subset t^{-1}C_{k-1}$ and the special case of the lemma, proved above, it follows that we can construct a sequence of lattices, joining C_{k-1} and C_k for some $\alpha_k \geqq 0$ with required properties. Amalgamating these sequences in a chain, we shall get the statement of the lemma.

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